The Measurement of Consumer Surplus.

PRIMAL PROBLEM

UTILITY MAXIMIZATION

Optimization Problem

$$\max \quad U(x_i, \ldots, x_n)$$
$$\text{s.t.} \quad \sum_i p_i x_i = m$$

Lagrangean

$$L = U(x_i, \ldots, x_n) + \lambda \left(m - \sum_i p_i x_i \right)$$

Optima - First-order conditions

$$\frac{\partial L}{\partial x_i} = \frac{\partial U}{\partial x_i} - \lambda p_i = 0 \qquad (1)$$
$$\frac{\partial L}{\partial \lambda} = m - \sum_i p_i x_i = 0 \qquad (2)$$

Optima - Second-order conditions

$$L_{ij} = \frac{\partial^2 L}{\partial x_i \partial x_j} \qquad L_{i\lambda} = \frac{\partial^2 L}{\partial x_i \partial \lambda}$$

$$D = \begin{vmatrix} L_{11} & L_{12} & \cdots & L_{1n} & L_{1\lambda} \\ L_{21} & L_{22} & \cdots & L_{2n} & L_{2\lambda} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ L_{n1} & L_{n2} & \cdots & L_{nn} & L_{n\lambda} \\ L_{\lambda 1} & L_{\lambda 2} & \cdots & L_{\lambda n} & L_{\lambda\lambda} \end{vmatrix} > 0 \qquad (3)$$

Uncompensated demand functions

(1) and (2) represent (n + 1) equations in (n + 1) unknowns (the x_i's and λ). Implicit function theorem (applicable because (3) is satisfied) allows the representation of money income held constant demand curves:

$$x_i = x_i^* \ (p, m)$$
$$\lambda = \lambda^* \ (p, m) \qquad (4)$$

DUAL PROBLEM

EXPENDITURE MINIMIZATION

Optimization Problem

$$\min \quad \sum_i p_i x_i$$
$$\text{s.t.} \quad U(x_1, \ldots x_n) = \overline{U}$$

Lagrangean

$$L = \sum_i p_i x_i + \mu \left(\overline{U} - U[x_1, \ldots x_n] \right)$$

Optima - First-order conditions

$$\frac{\partial L}{\partial x_i} = p_i - \mu \frac{\partial U}{\partial x_i} = 0 \qquad (1')$$
$$\frac{\partial L}{\partial \mu} = \overline{U} - U(x_i, \ldots x_n) = 0 \qquad (2')$$

Optima - Second-order conditions

$$D = \begin{vmatrix} L_{ij} \end{vmatrix} < 0 \qquad (3')$$

Compensated demand functions

(1) and (2) represent (n + 1) equations in (n + 1) unknowns (the x_i's and μ). Implicit function theorem (applicable because (3') is satisfied) allows the representation of utility held constant demand curves:

$$x_i = x_i^c(p, u)$$
$$\mu = \mu^c(p, u) \qquad (4')$$

PRIMAL PROBLEM

INDIRECT UTILITY FUNCTION

Substitute (4) into the optimization problem, so that

$$V(p_i, m) = U\left(x_1[p_i, m], \ldots, x_n[p_i, m]\right)$$

$$\text{s.t.} \quad \sum_i p_i x_i = m$$

V ( ) is the indirect utility function.

Uncompensated demand functions

$$\frac{\partial V}{\partial p_i} = \sum_j \frac{\partial U}{\partial x_j} \frac{\partial x_j}{\partial p_i}$$

$$= \sum_j \left(\lambda p_j\right) \frac{\partial x_j}{\partial p_i} \qquad \text{(by first-order condition)}$$

$$= \lambda \sum_j p_j \frac{\partial x_j}{\partial p_i}$$

The above expression can be further simplified by using the budget constraint.

$$\frac{\partial}{\partial p_i}\left[\sum_i p_i x_i\right] = \frac{\partial m}{\partial p_i} = 0$$

$$x_i + \sum_j p_j \frac{\partial x_j}{\partial p_i} = 0$$

$$\sum_j p_j \frac{\partial x_j}{\partial p_i} = -x_i$$

Substitute this expression into the expression for $\partial V / \partial p_i$ to get

$$\frac{\partial V}{\partial p_i} = -\lambda x_i \qquad (5)$$

What is the value of λ?

$$\frac{\partial V}{\partial m} = \sum_i \frac{\partial U}{\partial x_i} \frac{\partial x_i}{\partial m}$$

$$\frac{\partial V}{\partial m} = \lambda \sum_i p_i \frac{\partial x_i}{\partial m}$$

DUAL PROBLEM

EXPENDITURE FUNCTION

Substitute (4') into the optimization problem, so that

$$e(p_i, u) = \sum_i p_i x_i^c(p_i, u)$$

$$\text{s.t.} \quad U(x_i) = \overline{U}$$

e ( ) is the expenditure function.

Compensated demand functions

$$\frac{\partial e}{\partial p_i} = x_i^c + \sum_j p_j \frac{\partial x_j^c}{\partial p_i} \qquad (5')$$

By the utility constraint,

$$\frac{\partial}{\partial p_i}\left[U(x_i)\right] = \frac{\partial}{\partial p_i}\left[\overline{U}\right]$$

$$\sum_j \frac{\partial U}{\partial x_j} \frac{\partial x_j^c}{\partial p_i} = 0 \qquad (6')$$

$$\frac{\partial U}{\partial x_j} = \frac{1}{\mu} p_j \qquad \text{(by first-order condition)}$$

Substituting into (6')

$$\frac{1}{\mu} \sum_j p_j \frac{\partial x_j^c}{\partial p_i} = 0$$

If marginal utility of income is non-zero, then $\mu \neq 0$.

$$\sum_j p_j \frac{\partial x_j^c}{\partial p_i} = 0$$

Substituting this result into (5') gives

$$\frac{\partial e}{\partial p_i} = x_i^c \qquad \text{(7'; Hotelling's Lemma)}$$

PRIMAL PROBLEM	DUAL PROBLEM
By the budget constraint, $$\frac{\partial}{\partial m}\left(\sum_i p_i x_i\right) = \frac{\partial m}{\partial m} = 1$$ $$\sum_i p_i \frac{\partial x_i}{\partial m} = 1$$ $$\therefore \frac{\partial V}{\partial m} = \lambda \qquad (6)$$ Substitute (6) into (5): $$\frac{\partial V}{\partial p_i} = -\frac{\partial V}{\partial m} x_i$$ $$-\frac{\partial V / \partial p_i}{\partial V / \partial m} = x_i \qquad \text{(7; Roy's Identity)}$$ (7) gives another representation of the uncompensated demand functions; they reflect price and income effects on the indirect utility function.	7' are again the compensated demand functions; they reflect <u>only</u> price effects on the expenditure function. These demand curves are different from the uncompensated demand curves, which reflect price <u>and</u> income effects.
<u>Measuring welfare change</u> The measure of welfare change is obtained by differentiation of the indirect utility function: $$dV = \sum_i \frac{\partial V}{\partial p_i} dp_i + \frac{\partial V}{\partial m} dm$$ Substituting the results of equations (5) and (6), $$dV = -\lambda \sum_i x_i \, dp_i + \lambda \, dm$$ $$dW = \frac{dV}{\lambda} = -\sum_i x_i \, dp_i + dm$$ For a discrete change in price, $$\Delta W = -\sum_i \int_{p_i^1}^{p_i^2} x_i(p, m) dp_i + \Delta m \qquad (8)$$	<u>Measuring welfare change</u> The measure of welfare change is obtained by differentiating the expenditure function: $$de = \sum_i \frac{\partial e}{\partial p_i} dp_i + \frac{\partial e}{\partial u} du$$ $$de = \sum_i x_i^c \, dp_i + \mu \, du$$ This expression can be evaluated only by choosing a particular value of U, so that du = 0. If we choose initial utility, we associate welfare change with the compensating variation (CV). CV is the expenditure change needed to maintain initial utility at the new prices

PRIMAL PROBLEM	**DUAL PROBLEM**
X_i is a function of p and m, and both are changing simultaneously. Unless $\dfrac{\partial x_i}{\partial p_j} = \dfrac{\partial x_j}{\partial p_i}$ for all i and j, the value of this integral is not unique. Instead, it depends on the sequence of price changes. This problem is known as *path dependence*.	$CV = e(p^2, u^1) - e(p^1, u^1)$ $= \sum_i \int_{p_i^1}^{p_i^2} \dfrac{\partial e}{\partial p_i} dp_i$ $= \sum_i \int_{p_i^1}^{p_i^i} x_i(p, u)\, dp_i$ This measure is *path-independent*, because $\dfrac{\partial x_i}{\partial p_j} = \dfrac{\partial^2 e}{\partial p_i \partial p_j} = \dfrac{\partial^2 e}{\partial p_j \partial p_i} = \dfrac{\partial x_j}{\partial p_i}$ (from 7', Hotelling's Lemma).