Indeterminacy of Rate of Change of Marginal Utility of Income

This handout examines the implications of utility maximization for the marginal utility of income. Utility maximization is represented as follows:

$$\max \ U\left(x^1, x^2, \ldots, x^n\right) \quad \text{subject to} \quad \sum_i p^i x^i = Y$$

$$L = U\left(x^i\right) + \lambda\left(Y - \sum_i p^i x^i\right) \qquad \lambda \text{ is the marginal utility of income, } \partial U/\partial Y.$$

The first order conditions are as follows:

$$\frac{\partial L}{\partial x^i} = \frac{\partial U}{\partial x^i} - \lambda p^i = 0, \ \forall i \qquad (1)$$

To calculate the rate of change of the marginal utility of income, differentiate the first-order conditions with respect to Y:

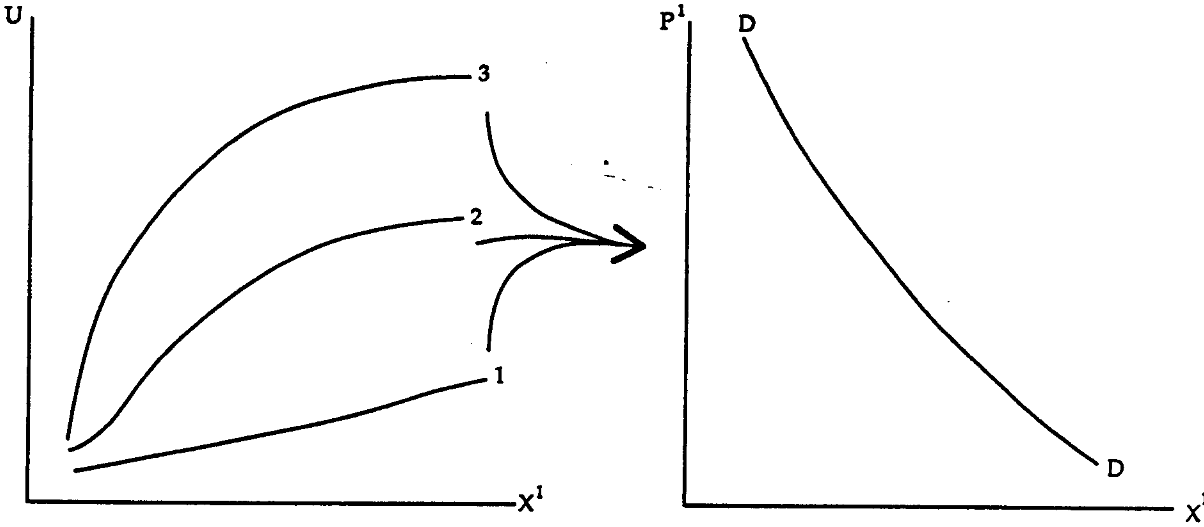
$$\frac{\partial}{\partial Y}\left(\frac{\partial U}{\partial x^i}\right) = p^i \frac{\partial \lambda}{\partial Y} \rightarrow \frac{\partial^2 U}{\partial x^i x^1}\frac{\partial x^1}{\partial Y} + \frac{\partial^2 U}{\partial x^i x^2}\frac{\partial x^2}{\partial Y} + \ldots = p^i \frac{\partial \lambda}{\partial Y}$$

$$\text{Let} \quad U_{ij} = \frac{\partial^2 U}{\partial x^i x^j}$$

Then

$$\sum_j U_{ij}\frac{\partial x^j}{\partial Y} = p^i \frac{\partial \lambda}{\partial Y} \qquad \therefore \frac{\partial \lambda}{\partial Y} \gtreqless 0 \ \text{as} \ \sum_j U_{ij}\frac{\partial x_j}{\partial Y} \gtreqless 0 \qquad (2)$$

A popular assumption is that $\frac{\partial \lambda}{\partial Y} < 0$ (diminishing marginal utility of income). We never observe utility functions, only demand functions for price-quantity and income-quantity relationships. It happens that identical demand functions are generated from utility functions that are monotonic transformations of one another:



What happens to the value of $\partial\lambda/\partial Y$ under a monotonic transformation of the utility function?

$$\max \quad V\left(U\left[x^i\right]\right) \quad \text{s.t.} \quad \sum_i p^i x^i = Y$$

$$L = V\left(U\left[x^i\right]\right) + \gamma\left(Y - \sum_i p^i x^i\right)$$

First order conditions:

$$\frac{dV}{dU}\frac{\partial U}{\partial x^i} - \gamma p^i = 0 \quad \forall i$$

$$\rightarrow V' \frac{\partial U}{\partial x^i} = \gamma p^i \rightarrow \gamma = V' \lambda \qquad \text{(from eq. 1 above)} \qquad (3)$$

Calculate the rate of change of MU of income for the transformed utility function:

$$\frac{\partial}{\partial Y}\left[V' \frac{\partial U}{\partial x^i}\right] = \frac{\partial}{\partial Y}\left[V' \lambda p^i\right]$$

$$\rightarrow V' \frac{\partial}{\partial Y}\left[\frac{\partial U}{\partial x^i}\right] + \frac{\partial U}{\partial x^i}\left[V''\left(\frac{\partial U}{\partial x^1}\frac{\partial x^1}{\partial Y} + \frac{\partial U}{\partial x^2}\frac{\partial x^2}{\partial Y} + \ldots\right)\right] = p^i\left[\frac{\partial V'}{\partial Y}\lambda + V'\frac{\partial \lambda}{\partial Y}\right]$$

$$\rightarrow V'\left[\sum_j U_{ij}\frac{\partial x^j}{\partial Y}\right] + U_i V'' \sum_j U_j \frac{\partial x^j}{\partial Y} = p^i\left[\lambda V'' \sum_j U_j \frac{\partial x^j}{\partial Y} + V' \frac{\partial \lambda}{\partial Y}\right], \qquad (4)$$

where the subscripted U's indicate partial derivatives.

From (2) above, if $\frac{\partial \lambda}{\partial Y} < 0$, $\sum_j U_{ij}\frac{\partial x^j}{\partial Y} < 0$

By the assumption of a monotonic transformation, $V' > 0$

But equation (4) shows that the new marginal utility of income,

$$\frac{\partial \gamma}{\partial Y} < 0 \quad \text{iff} \quad V' \frac{\partial \lambda}{\partial Y} + \lambda V'' \sum_j U_j \frac{\partial x^j}{\partial Y} < 0$$

This means, for the assumed $\partial\lambda/\partial Y < 0$, a large positive value of the term $\lambda V'' \sum_j U_j \frac{\partial x^j}{\partial Y}$ could provide a $\partial\gamma/\partial Y > 0$ a positive rate of change of the marginal utility of income. This can be achieved by choosing $V'' > 0$ (and large). The monotonically-transformed utility function could have a rate of change in the marginal utility of income opposite to that of the original utility function. Therefore, observed demand behavior can correspond with $\frac{\partial \lambda}{\partial Y} > 0$ (increasing MU of income) OR $\frac{\partial \lambda}{\partial Y} < 0$ (diminishing MU).