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Ag Econ 514/Econ 514/MAP 514

Extinction

Imagine a single commodity (x) economy, with the amount consumed by generation t represented as x_t. Assume the stock of the commodity at time t is X_t. The amount left to the generation (t + 1) is then $\left(X_{\bar{t}} - x_t\right)$. The commodity has a natural growth rate of g, so that total availability for generation (t + 1) is estimated as follows:

$$X_{t+1} = (1 + g)\left(X_t - x_t\right) \qquad (1)$$

Assume a social welfare function that is a function of the utility of consumption of each generation:

$$W = W\left(U(x_1),\ U(x_2),\ U(x_3)\ldots\right)$$

Assume further that each generation has the same utility function, with diminishing marginal utility of consumption.

To maximize social welfare, we try to maximize the discounted present value of all the generation's consumption:

$$\max \quad \sum_{t=0}^{\infty} U(x_t)\frac{1}{(1+r)^t} \qquad (2)$$

$$\begin{aligned}
\text{subject to} \quad & X_{t+1} = (1+g)\left(X_t - x_t\right) \\
& X_t \geq 0 \\
& x_t \geq 0 \\
& X_0 = \overline{X}_0
\end{aligned}$$

Form the Lagrangean:

$$L = \sum_{t=0}^{\infty}\left(\frac{U(x_t)}{(1+r)^t}\right) + \lambda_t\left[(1+g)\left(X_t - x_t\right) - X_{t+1}\right] + \lambda_0\left(\overline{X}_0 - X_0\right) \qquad (3)$$

Optimality conditions are:

$$\frac{\partial L}{\partial x_t} = U'\left(x_t\right)(1+r)^{-t} - \lambda_t(1+g) = 0 \qquad (4)$$

$$\frac{\partial L}{\partial X_t} = \lambda_t(1+g) - \lambda_{t-1} = 0 \qquad (5)$$

Equation (5) can be rearranged to show $\lambda_{(t-1)} = \lambda_t (1 + g)$

So λ must decline over time at the natural growth rate:

$$\lambda_t = \lambda_0 (1 + g)^{-t} \qquad (6)$$

Substitute (6) in (4) to get another expression for the optimum:

$$\frac{U'(x_t)}{(1 + r)^t} = \lambda_0 (1 + g)^{-t} \qquad (7)$$

$$U'(X_t) = \lambda_0 \left[\frac{(1 + r)}{(1 + g)}\right]^t \qquad (8)$$

If the discount rate is greater than the growth rate, then the value of the marginal utility of income must be increasing over time. This is possible only if consumption is decreasing over time, implying extinction.